

Noncommutative Gravity Solutions

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Abstract

We consider noncommutative geometries obtained from a triangular Drinfeld twist and review the formulation of noncommutative gravity. A detailed study of the abelian twist geometry is presented, including the fundamental theorem of noncommutative Riemannian geometry. Inspired by [1, 2], we obtain solutions of noncommutative Einstein equations by considering twists that are compatible with the curved spacetime metric.

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1 Introduction

The study of solutions of noncommutative field theories is of primary importance for a better understanding of the theories themselves and in discussing their phenomenological implications. Some of the relevant literature on noncommutative gravity solutions can be found in references [3], [4]. Of particular interest are noncommutative black hole solutions and noncommutative cosmological solutions. The study of solutions may also help to understand the relations between different approaches to noncommutative gauge and gravity theories. In the case of noncommutative gravity we mention for example the metric approach of ref.s [5, 6] and the vielbein approach that allows coupling to fermions of ref.s [7, 8].

We here study solutions of the noncommutative Einstein equations considered in [5, 6].

The theory in [5, 6] is obtained by demanding covariance under noncommutative general coordinate transformations and it is based on noncommutative Riemannian geometry. It can address local and also global (topological) aspects in noncommutative general relativity and noncommutative geometry. The theory therefore allows also for a detailed study of its solutions that goes beyond a qualitative analysis. This is object of the present paper.

Noncommutative Einstein equations are in principle quite hard to solve. By expanding these equations in the noncommutativity parameter θ we see that they contain partial derivatives of any order, the order increasing with the power of θ . One can consider approximate gravity solutions by truncating this expansion at a given order in θ . This is physically sensible if we consider θ as a small perturbation to commutative spacetime. On the other hand it is also important to present exact nonperturbative solutions. Here one approach, that has been fruitfully studied in noncommutative gauge theories, is to consider topological solutions, like for example gravitational instantons solutions. Our approach, inspired by [1, 2], is to consider noncommutative gravity solutions with symmetry properties.

In the framework of [5, 6], that formulates noncommutative gravity based on Drinfeld twists [9], we prove that, provided the twist is in part constructed with Killing vector fields, undeformed gravity solutions are also noncommutative gravity solutions. In this paper we study the case of Einstein equations in vacuum with and without cosmological constant, i.e., we study Einstein spaces. Our results holds for metrics with arbitrary signature, in particular Euclidean and Lorentian. We stress that the commutative and noncommutative Einstein equations are different and therefore the corresponding set of solutions are different. It is only by requiring compatibility between the twist and the metric that we are able to find a subset of solutions that is common to both the commutative and the noncommutative theories. Note that even in this case noncommutative and commutative connections in general differ, and hence geodesic motions also differ. It is interesting to investigate the physical implications of these differences.

The class of star products and noncommutative manifolds we consider is a rather large class. The examples in Section 3 include quantization of symplectic and also of Poisson structures. The algebra of functions of the noncommutative torus, of the noncommutative spheres [10] and of further noncommutative manifolds (so-called isospectral deformations) considered in [10], and in [11], [12], is associated to a \star -product structure obtained via a triangular Drinfeld twist (see [14] and, for the four-sphere in [10], see [13], [15]). In these cases the twist is abelian and entirely constructed with Killing vector fields, we consider the more general case where the triangular Drinfeld twist only in part contains Killing vector fields (cf. eq. (3.19), (4.25)). The star products we study are however not the most general ones, in particular they are a subclass of those associated with a quasitriangular structure [16]: on that noncommutative algebra of functions there is an action of the braid

group, while in the triangular case there is an action of the permutation group.

It is remarkable how far in the program of formulating a noncommutative differential geometry one can go using triangular Drinfeld twists. The study of this class of \star -products geometries are first examples that can uncover some common features of a wider class of noncommutative geometries.

We consider three kinds of twists, in order of increasing generality:

I. Moyal-Weyl twist (associated with the Moyal-Weyl \star -product),

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}$$

II. Abelian twist (socalled because the vectorfields $\{X_a\}$ are mutually commuting),

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a\otimes X_b}$$

III. General (triangular) Drinfeld twist \mathcal{F} .

In Section 2 we recall the main results of metric noncommutative gravity with Moyal-Weyl twist. Particular solutions are discussed in Section 3. They are found by listing all the metrics \mathbf{g} compatible with the twist \mathcal{F} , in the sense that the twist \mathcal{F} is constructed in part with Killing vector fields of \mathbf{g} . Among these metrics those that are classically Einstein metrics are also shown to be noncommutative Einstein metrics.

In Section 4 we study the Riemannian geometry of abelian twists on a smooth manifold M . We locally reduce this study to the Riemannian geometry of the Moyal-Weyl twist and then properly glue the local results in order to obtain the global ones on the whole noncommutative manifold M . Here too if the twist \mathcal{F} contains in part Killing vector fields of a commutative Einstein metric \mathbf{g} then \mathbf{g} is also a noncommutative Einstein metric. The corresponding noncommutative Levi-Civita connection is also given, and it is different from the commutative one.

The differential and Riemannian geometry of a general triangular Drinfel twist is recalled in Section 5. New results include a detailed study of the contraction operator. Uniqueness of the Levi-Civita connection of a noncommutative (pseudo-)Riemannian manifold is also proven. Finally in Section 6 twisted gravity solutions are considered for general Drinfeld twists (case III above) that are constructed with affine Killing vectors. Here, differently from Section 3 and Section 4 (case I and II above), on one hand we require a stronger compatibility condition between the twist and the metric, the twist being fully (and not in part) constructed with affine Killing vectors; on the other hand we relax the Killing condition by considering the wider class of affine or homotetic Killing vectors.

2 Gravity with Moyal-Weyl \star -noncommutativity

Functions. The star product that implements the $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$ noncommutativity is given by

$$(h \star g)(x) = e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} h(x)g(y)|_{x=y} \quad (2.1)$$

where h and g are arbitrary functions. This star product between functions can be obtained from the usual pointwise product $(hg)(x) = h(x)g(x)$ via the action of a twist operator \mathcal{F}

$$h \star g := \mu \circ \mathcal{F}^{-1}(h \otimes g) , \quad (2.2)$$

where μ is the usual pointwise product between functions, $\mu(f \otimes g) = fg$, and the twist operator and its inverse are

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} , \quad \mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} . \quad (2.3)$$

We shall frequently use the notation (sum over α understood)

$$\mathcal{F} = f^\alpha \otimes f_\alpha , \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha , \quad (2.4)$$

so that

$$f \star g := \bar{f}^\alpha(f) \bar{f}_\alpha(g) . \quad (2.5)$$

In [5] we developed a noncommutative geometry based on the twist (2.3) and the associated Moyal-Weyl \star -product (see also [17, 18]). The twist allows to define \star -products between functions and vector fields and more in general between tensor fields. We thus obtained a deformed differential and Riemannian geometry that we briefly summarize.

Vector fields. Partial derivatives act on vector fields $v = v^\nu \partial_\nu$ via the Lie derivative action

$$\mathcal{L}_{\partial_\mu}(v) = [\partial_\mu, v] = \partial_\mu(v^\nu) \partial_\nu . \quad (2.6)$$

Similarly to (2.5) the product $\mu : A \otimes \Xi \rightarrow \Xi$ between the space A of functions and the space Ξ of vector fields is deformed into the product

$$h \star v = \bar{f}^\alpha(h) \bar{f}_\alpha(v) , \quad (2.7)$$

where \bar{f}_α acts on vectors via the Lie derivatives $\mathcal{L}_{\bar{f}_\alpha}$ (Lie derivative along products of elements of the Lie algebra of vector fields Ξ are defined simply by $\mathcal{L}_{uv\dots} = \mathcal{L}_u \mathcal{L}_v \dots$). Since $\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu} \partial_\mu \otimes \partial_\nu}$, we obtain

$$h \star \partial_\rho = h \partial_\rho \quad (2.8)$$

This is so because the twist \mathcal{F} acts trivially on the vector field ∂_ν (cf. (2.6)). More in general we have $h \star v = \bar{f}^\alpha(h) \bar{f}_\alpha(v) = \bar{f}^\alpha(h) \bar{f}_\alpha(v^\rho) \partial_\rho = (h \star v^\rho) \partial_\rho$.

We denote the space of vector fields with this \star -multiplication by Ξ_\star . As vector spaces $\Xi = \Xi_\star$, but Ξ is an A -module while Ξ_\star is an A_\star -module.

Tensor fields. Tensor fields form an algebra with the tensor product \otimes (over the algebra of functions). We define \mathcal{T}_\star to be the noncommutative algebra of tensor fields. As vector spaces $\mathcal{T} = \mathcal{T}_\star$; the noncommutative and associative tensor product is obtained as in (2.5) and (2.7):

$$\tau \otimes_\star \tau' := \bar{f}^\alpha(\tau) \otimes \bar{f}_\alpha(\tau') . \quad (2.9)$$

here too, as in (2.7), \bar{f}^α and \bar{f}_α act on tensors via the Lie derivatives $\mathcal{L}_{\bar{f}^\alpha}$ and $\mathcal{L}_{\bar{f}_\alpha}$. Notice that the action of the twist on the one forms dx^ρ is trivial

$$\tau \otimes_\star dx^\rho = \tau \otimes dx^\rho , \quad dx^\rho \otimes_\star \tau' = dx^\rho \otimes \tau' , \quad (2.10)$$

in particular $h \star dx^\rho = dx^\rho \star h = h dx^\rho$. This is so because the Lie derivative along the vectors ∂_ν entering the twist \mathcal{F} vanishes on dx^ρ , $\mathcal{L}_{\partial_\nu} dx^\rho = 0$.

Vector fields act on tensor fields via the Lie derivative, and they form the Lie algebra of infinitesimal (local) diffeomorphisms. The space of \star -vector fields Ξ_\star has similarly a \star -Lie derivative action on tensor fields. We have a deformed Leibniz rule so that if τ and τ' transform as tensors also $\tau \otimes_\star \tau'$ transforms as a tensor (and the matrix $\theta^{\mu\nu}$ remains invariant). Correspondingly we have a \star -Lie algebra of vector fields or of deformed infinitesimal (local) diffeomorphisms.

A covariant derivative ∇_μ^\star that has the same Leibnitz rule as the \star -Lie derivative can then be naturally constructed.

The \star -differential geometry formulae simplify if we use the basis ∂_μ and the dual basis dx^μ . Following [5] (see also [17]) for any (undeformed) metric tensor

$$\mathbf{g} = \mathbf{g}_{\mu\nu} dx^\mu \otimes dx^\nu = dx^\mu \otimes_\star dx^\nu \star \mathbf{g}_{\mu\nu} \quad (2.11)$$

there exists a unique metric compatible and torsionfree covariant derivative ∇_u^\star . The Christoffel symbols are defined by

$$\nabla_\mu^\star \partial_\rho = \Gamma_{\mu\rho}^{\star\ \nu} \star \partial_\nu , \quad (2.12)$$

or equivalently by $\nabla_\mu^\star dx^\nu = -dx^\rho \star \Gamma_{\mu\rho}^{\star\ \nu}$. A derivation similar to the one of the undeformed case leads to the explicit expression

$$\Gamma_{\mu\nu}^{\star\ \rho} = \frac{1}{2} (\partial_\mu \mathbf{g}_{\nu\sigma} + \partial_\nu \mathbf{g}_{\sigma\mu} - \partial_\sigma \mathbf{g}_{\mu\nu}) \star \mathbf{g}^{\star\sigma\rho} , \quad (2.13)$$

where $\mathbf{g}^{\star\sigma\rho}$ is the \star -inverse metric

$$\mathbf{g}^{\star\sigma\rho} \star \mathbf{g}_{\rho\nu} = \delta_\nu^\sigma , \quad \mathbf{g}_{\nu\rho} \star \mathbf{g}^{\star\rho\sigma} = \delta_\nu^\sigma . \quad (2.14)$$

The torsion tensor is $\mathsf{T}_{\mu\nu}^{\star\rho} = \Gamma_{\mu\nu}^{\star\rho} - \Gamma_{\nu\mu}^{\star\rho}$ and vanishes. The curvature tensor is given by¹

$$\mathsf{R}_{\rho\mu\nu}^{\star\sigma} = \partial_\mu \Gamma_{\nu\rho}^{\star\sigma} - \partial_\nu \Gamma_{\mu\rho}^{\star\sigma} + \Gamma_{\nu\rho}^{\star\beta} \star \Gamma_{\mu\beta}^{\star\sigma} - \Gamma_{\mu\rho}^{\star\beta} \star \Gamma_{\nu\beta}^{\star\sigma} . \quad (2.15)$$

As in the commutative case the Ricci tensor is a contraction of the curvature tensor,

$$\mathsf{Ric}_{\mu\nu}^{\star} = \mathsf{R}_{\mu\rho\nu}^{\star\rho} . \quad (2.16)$$

Finally the noncommutative version of Einstein equation in vacuum and in the presence of a cosmological constant $c \in \mathbb{R}$ is

$$\mathsf{Ric}_{\mu\nu}^{\star} = c g_{\mu\nu} . \quad (2.17)$$

In [6] we defined connection, curvature and Ricci curvature in a more intrinsic geometric language (without using coordinates) and in the case of an arbitrary manifold with noncommutativity given by a Drinfeld twist. As we review in Section 5.5 the Einstein equation in vacuum and in the presence of a cosmological constant reads

$$\mathsf{Ric}^{\star} = c g . \quad (2.18)$$

Metrics g that satisfy (2.18) are called \star -Einstein metrics, and (M, \mathcal{F}, g) is a \star -Einstein space. Of course, if there exist local coordinates where the twist assumes the canonical Moyal-Weyl form (2.3), then, in the open domain of the chart defined by these coordinates, the connection, curvature and Ricci curvature are equivalently given by expressions (2.13), (2.15), (2.16); and (2.18) is equivalent to (2.17).

3 Gravity solutions I: Moyal-Weyl \star -product

In this section we consider the manifold \mathbb{R}^4 (or \mathbb{R}^N) with canonical Moyal-Weyl \star -product, or equivalently we work locally in an open neighbourhood with coordinates $\{x^\mu\}$ that satisfy the Moyal-Weyl relations $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$ (the constant matrix $\theta^{\mu\nu}$ being possibly degenerate).

The class of noncommutative (NC) Einstein metrics we consider are obtained by selecting those Einstein metrics of commutative spacetime that are compatible with the twist.

Theorem 1. Given the twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$ and a metric g ,

i) If the Killing Lie algebra g_K of the metric g has the twist compatibility property

$$\theta^{\mu\nu} \partial_\mu \otimes \partial_\nu \in \Xi \otimes g_K + g_K \otimes \Xi \quad (3.19)$$

¹The relation between the coefficients $\mathsf{R}_{\mu\nu\rho}^{\star\sigma}$ defined in [17, 18] and those in (2.15) is $\mathsf{R}_{\mu\nu\rho}^{\star\sigma} = \mathsf{R}_{\rho\mu\nu}^{\star\sigma}$.

then the NC curvature and the NC Ricci curvature of the NC Levi-Civita connection are the undeformed ones.

ii) If the metric g is an Einstein metric then it is also a NC Einstein metric.

Proof. *i)* We show that the star product disappears from the expressions for the NC curvature, Ricci tensor and Einstein equation. The inverse metric g^{-1} is invariant under the same Lie algebra g_K as g and hence coincides with the \star -inverse metric, $g^{\sigma\rho} \star g_{\rho\nu} = g^{\sigma\rho} g_{\rho\nu} = \delta_\nu^\sigma$ because either the left or the right leg of (3.19) acts trivially. Similarly the \star -product drops out from the Christoffel symbols (2.13) and from the curvature expression (2.15). The NC Einstein equation reduces then to the commutative one, and property *ii)* follows. \square

We remark that the NC covariant derivative ∇_μ^\star differs from the undeformed one ∇_μ . This is due to the different Leibniz rule,

$$\nabla_\mu^\star(h \star \partial_\rho) = \partial_\mu h \star \partial_\rho + h \star \Gamma_{\mu\rho}^{\star\ \nu} \star \partial_\nu = \partial_\mu h \partial_\rho + h \star \Gamma_{\mu\rho}^{\ \nu} \partial_\nu \quad (3.20)$$

$$\nabla_\mu(h \star \partial_\rho) = \nabla_\mu(h \partial_\rho) = \partial_\mu h \partial_\rho + h \Gamma_{\mu\rho}^{\ \nu} \partial_\nu \neq \nabla_\mu^\star(h \star \partial_\rho) \quad (3.21)$$

The covariant derivative ∇_μ^\star can consistently be extended along any vector field $u = u^\mu \partial_\mu = u^\mu \star \partial_\mu$ by defining, for all $v \in \Xi$

$$\nabla_u^\star v = u^\mu \star \nabla_\mu^\star v. \quad (3.22)$$

This introduces another source of difference, $\nabla_u^\star v = u^\mu \star \nabla_\mu^\star v \neq \nabla_u v$. Because of (3.20) and of (3.22), NC geodesic motion is expected to differ from the undeformed one.

4 Geometry and Gravity solutions II: Abelian Twist

We generalize the results of the previous section to the case of any manifold M with abelian twist, i.e., with a twist \mathcal{F} of the form

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}, \quad (4.23)$$

where the vector fields $\{X_a\}$ are mutually commuting $[X_a, X_b] = 0$. This property implies the associativity of the corresponding \star -product between functions (2.2), and between tensors (2.9).

The abelian twist case can be reduced to the previous Moyal-Weyl twist case. Let

$$|\text{span}\{X_a\}|_P \quad (4.24)$$

be the dimension of the vector space spanned by the commuting vector fields $\{X_a\}$ at point $P \in M$.

We call P a regular point of the twist \mathcal{F} if there is an open neighbourhood of P where $|\text{span}\{X_a\}|$ is constant. Notice that this open neighborhood is itself made of regular points, and hence the set of all regular points is an open submanifold of M . We denote it by M_{reg} .

If P is a regular point we can always write $\theta^{ab}X_a \otimes X_b = \tilde{\theta}^{\tilde{a}\tilde{b}}\tilde{X}_{\tilde{a}} \otimes \tilde{X}_{\tilde{b}}$ where the range of the index \tilde{a} is not greater than that of a and $\tilde{X}_{\tilde{a}}$ are linearly independent vectors in each point of the open neighbourhood of P where $|\text{span}\{X_a\}|$ is constant.

It follows that locally around P we can consider coordinates $\{x^\mu\} = \{x^{\tilde{a}}, x^i\}$ (where $i = 1, \dots, \dim M - |\text{span}\{X_a\}|$) such that $\tilde{X}_{\tilde{a}} = \frac{\partial}{\partial x^{\tilde{a}}}$ (Frobenius theorem). In this coordinate system the twist \mathcal{F} has the canonical Moyal-Weyl structure $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$. We therefore have a Levi-Civita connection defined via its Christoffel symbols, and we can also apply Theorem 1. This holds for any regular point $P \in M$. These local properties glue together to give the corresponding global ones on M_{reg} .

Theorem 2. Given a manifold M with abelian twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}$, consider the manifold M_{reg} of regular points of \mathcal{F} and consider a metric \mathbf{g} of M_{reg} .

- i) There exists a unique NC Levi-Civita connection on M_{reg} associated with the metric \mathbf{g} .
- ii) If the Killing Lie algebra g_K of the metric \mathbf{g} has the twist compatibility property

$$\theta^{ab}X_a \otimes X_b \in \Xi \otimes g_K + g_K \otimes \Xi \quad (4.25)$$

then the NC curvature and the NC Ricci curvature of the NC Levi-Civita connection are the undeformed ones.

- iii) If the metric \mathbf{g} is an Einstein metric then it is also a NC Einstein metric.

Proof. i) The manifold M_{reg} has an atlas of charts where the twist assumes the canonical Moyal-Weyl form. Let U and \tilde{U} be the domains of charts with coordinates $\{x^\mu\}$ and $\{y^\epsilon\}$ respectively. We use the indices μ, ν, ρ for the x -coordinates in U , and the indices ϵ, ζ, κ for the y -coordinates in \tilde{U} . We show that if $U \cap \tilde{U} \neq \emptyset$ then the transition functions λ_ϵ^μ defined in $U \cap \tilde{U}$ by

$$\frac{\partial}{\partial y^\epsilon} = \lambda_\epsilon^\mu \star \frac{\partial}{\partial x^\mu} \quad (4.26)$$

are invariant under the action of the \star -product, i.e., for all $h \in A_\star$,

$$h \star \lambda_\epsilon^\mu = \lambda_\epsilon^\mu \star h = h \lambda_\epsilon^\mu. \quad (4.27)$$

Indeed, $h \lambda_\epsilon^\mu \partial_\mu = h \partial_\epsilon = h \star \partial_\epsilon = h \star (\lambda_\epsilon^\mu \star \partial_\mu) = (h \star \lambda_\epsilon^\mu) \star \partial_\mu = (h \star \lambda_\epsilon^\mu) \partial_\mu$, where we used that the twist acts trivially on the partial derivatives of both coordinate systems. The equality $\lambda_\epsilon^\mu \star h = h \lambda_\epsilon^\mu$ is similarly proven. [Hint: set $\tau = \partial_\mu$ and $\tau' = h$ in (2.9).]

We now prove that we have a globally defined connection on M_{reg} . Indeed the connections defined in U and in \tilde{U} by their Christoffel symbols (cf. (2.13)) coincide in the intersection $U \cap \tilde{U}$, for all $u \in \Xi$,

$$\nabla_u^\star = \tilde{\nabla}_u^\star. \quad (4.28)$$

Because of (3.22) this equality is proven if $\nabla_\varepsilon^\star = \tilde{\nabla}_\varepsilon^\star$ ($\varepsilon = 1, \dots, n = \dim M$). This is indeed the case, since

$$\begin{aligned} \nabla_\varepsilon^\star \partial_\zeta &= \lambda_\varepsilon^\mu \star \nabla_\mu^\star (\lambda_\zeta^\nu \star \partial_\nu) \\ &= \lambda_\varepsilon^\mu \star (\partial_\mu \lambda_\zeta^\nu \star \partial_\nu + \lambda_\zeta^\nu \star \Gamma_{\mu\nu}^\star{}^\rho \star \partial_\rho) \\ &= \lambda_\varepsilon^\mu (\partial_\mu \lambda_\zeta^\nu \lambda_\nu^\kappa + \lambda_\zeta^\nu \Gamma_{\mu\nu}^\star{}^\rho \lambda_\rho^\kappa) \partial_\kappa \\ &= \tilde{\Gamma}_{\varepsilon\zeta}^\star{}^\kappa \partial_\kappa \\ &= \tilde{\nabla}_\varepsilon^\star \partial_\zeta \end{aligned} \quad (4.29)$$

where in the third line we used property (4.27), and in the fourth line we used that the noncommutative Christoffel symbols transform under (4.26) as in the undeformed case. This holds because the metric and its \star -inverse transform as in the undeformed case (use associativity of the \star -product and (4.27)), and the transition functions λ_ε^μ , their inverses λ_μ^ζ and their partial derivatives are unaffected by the \star -product that appears in the NC Christoffel symbols.

ii) The NC curvature and Ricci tensors on M_{reg} of the globally defined NC Levi-Civita connection are the undeformed ones because equation (4.25) implies that the hypothesis (3.19) of Theorem 1 holds in an open neighbourhood of any regular point P .

Property *iii)* follows immediately from Theorem 1 and the observation that the Einstein metric condition (2.18) holds in M_{reg} if for any point $P \in M_{reg}$ there exists an open coordinate neighbourhood where the equivalent condition (2.17) holds. \square

In general $M_{reg} \neq M$, and actually the case $M_{reg} = M$ is a special one (in the semiclassical limit M is a regular Poisson manifold). A typical example where $M_{reg} \neq M$ is provided by the so-called Manin's plane or quantum plane. A star product that implements the quantum plane commutation relation $xy = qyx$ ($q = e^{i\hbar}$) can be obtained via the twist

$$\mathcal{F} = e^{-\frac{i}{2}\hbar(x\frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y} - y\frac{\partial}{\partial y} \otimes x\frac{\partial}{\partial x})}. \quad (4.30)$$

The vector fields are $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. The irregular points are the x and y coordinate axes. Another example is provided by the twists on group manifolds introduced by [19]. When the groups are nonabelian then there necessarily exist irregular points of the group manifold.

We now use continuity arguments to extend Theorem 2 from M_{reg} to all M . The first step is to show that (as for the regular points of a Poisson structure) the closure of M_{reg}

is M . The second step is to show that the NC connection has a unique extension from M_{reg} to M .

Theorem 3. The open submanifold M_{reg} is dense in M .

Proof. Let $n = \dim M$, and let R_n be the open submanifold of regular points P of M such that $|\text{span}\{X_a\}|_P = n$. The condition $|\text{span}\{X_a\}|_P = n$ is equivalent to require the matrix X of entries X_a^μ to have rank n . The entries X_a^μ are the coefficients of the vector fields X_a with respect to a local frame ∂_μ (and of course the rank is independent from the choice of the frame). In turn the rank n condition can be expressed by requiring the sum of the squares of all n^{th} -order minors² of the matrix X to be different from zero. We denote this function by $\sigma_{X,n}$. In formulae we have

$$|\text{span}\{X_a\}| = n \Leftrightarrow \sigma_{X,n} \neq 0. \quad (4.31)$$

Since the vector fields X_a are smooth, $\sigma_{X,n} : M \rightarrow \mathbb{R}$ is a smooth function, in particular it is a continuous function and therefore if $\sigma_{X,n}(P) \neq 0$ then there is a neighbourhood of P where $\sigma_{X,n} \neq 0$. Thus

$$\begin{aligned} R_n &\equiv \{P \in M; \text{there exists an open neighbourhood of } P \text{ where } |\text{span}\{X_a\}| = n\} \\ &= \{P \in M; \text{there exists an open neighbourhood of } P \text{ where } \sigma_{X,n} \neq 0\} \\ &= \{P \in M; \sigma_{X,n}(P) \neq 0\}. \end{aligned} \quad (4.32)$$

If $R_n = M$ the theorem is proven. If $R_n \neq M$ we denote by \overline{R}_n its closure and let R_{n-1} be the open submanifold of all regular points P of the complement \overline{R}_n' such that $|\text{span}\{X_a\}|_P = n - 1$. Proceeding as before we have

$$\begin{aligned} R_{n-1} &\equiv \{P \in \overline{R}_n'; \text{there exists an open neighbourhood of } P \text{ where } |\text{span}\{X_a\}| = n - 1\} \\ &= \{P \in \overline{R}_n'; \sigma_{X,n-1}(P) \neq 0\}. \end{aligned} \quad (4.33)$$

Notice that from the definition of R_{n-1} it follows $R_{n-1} \subset M_{reg}$ because \overline{R}_n' is open in M .

If $R_{n-1} = \overline{R}_n'$ then $R_{n-1} \cup R_n = \overline{R}_n' \cup R_n$ is dense in $\overline{R}_n' \cup \overline{R}_n = M$, and the theorem is proven. If $R_{n-1} \neq \overline{R}_n'$ then let \overline{R}_{n-1} be the closure in \overline{R}_n' of R_{n-1} , let \overline{R}_{n-1}' be its complement in \overline{R}_n' and let R_{n-2} be the open submanifold of all regular points P of \overline{R}_{n-1}' such that $|\text{span}\{X_a\}|_P = n - 2$.

If $R_{n-2} = \overline{R}_{n-1}'$ then $R_{n-2} \cup R_{n-1} = \overline{R}_{n-1}' \cup R_{n-1}$ which is dense in $\overline{R}_{n-1}' \cup \overline{R}_{n-1} = \overline{R}_n'$; therefore $R_{n-2} \cup R_{n-1} \cup R_n$ is dense in $\overline{R}_n' \cup R_n$ and hence dense in M because $\overline{R}_n' \cup R_n$ is dense in M (we use that if A is dense in B and B is open and dense in C then A is

²we recall that the n^{th} -order minors of a given matrix are the determinants of the $n \times n$ submatrices of the given matrix.

dense in C). If $R_{n-2} \neq \overline{R}'_{n-1}$ then we consider $R_{n-3} \subset \overline{R}'_{n-2}$ and so on (see Fig. 1). This iteration procedure stops at the latest when we consider R_0 .

If $R_0 = \overline{R}'_1$ the theorem is proven. We now show that $R_0 = \overline{R}'_1$ always holds. Indeed R_1 is the inverse image under $\sigma_{X,1} : \overline{R}'_2 \rightarrow \mathbb{R}$ of the open interval $\mathbb{R} - \{0\}$,

$$R_1 = \{P \in \overline{R}'_2; \sigma_{X,1}(P) \neq 0\} = \sigma_{X,1}^{-1}(\mathbb{R} - \{0\}) .$$

Hence

$$\begin{aligned} \overline{R}_1 &= \{P \in \overline{R}'_2; \text{ for all } I_P, I_P \cap \sigma_{X,1}^{-1}(\mathbb{R} - \{0\}) \neq \emptyset\} \\ &= \{P \in \overline{R}'_2; \text{ for all } I_P, \sigma_{X,1}(I_P) \neq \{0\}\} , \end{aligned}$$

where I_P is an open neighbourhood of P . It follows that

$$\overline{R}'_1 = \{P \in \overline{R}'_2; \text{ there exists an } I_P \text{ such that } \sigma_{X,1}(I_P) = \{0\}\} = R_0 .$$

□

We find practical to summarize the iteration procedure of Theorem 3 in Fig. 1.

$$\begin{array}{rcl} & & R_n \subset M \\ & & R_{n-1} \subset \overline{R}'_n \\ & & R_{n-2} \subset \overline{R}'_{n-1} \\ & \vdots & \vdots \\ & & R_1 \subset \overline{R}'_2 \\ R_0 & = & \overline{R}'_1 \end{array}$$

Fig. 1. The iteration procedure of Theorem 3.

In order to extend the connection from M_{reg} to M we recall (see [20]) that the covariant derivative ∇_u^* defines the connection ∇^* from vector fields to vector fields with values in 1-forms,³

$$\begin{aligned} \nabla^* : \Xi_\star &\rightarrow \Omega_\star \otimes_\star \Xi_\star \\ v &\mapsto \nabla^*(v) \end{aligned} \tag{4.34}$$

³in the commutative case we should write $\Gamma(TM \otimes T^*M)$ rather than the more intuitive expression $\Omega \otimes \Xi$. For smooth second countable finite dimensional manifolds M these two expressions coincide (see for ex. [21], Proposition 2.6). This follows from the existence of a finite covering of M that trivializes the tangent bundle TM and the cotangent bundle T^*M , see for example [22], theorem 7.5.16.

where

$$\begin{aligned}\nabla^*(v) : \Xi_\star &\rightarrow \Xi_\star \\ u &\mapsto \langle u, \nabla^*(v) \rangle_\star = \nabla_u^*(v)\end{aligned}\tag{4.35}$$

and the nondegenerate pairing $\langle \cdot, \cdot \rangle_\star$ between 1-forms $\omega \in \Omega_\star$ (or 1-forms with values in vector fields like $\nabla^*(v)$) and vector fields $u \in \Xi_\star$ is given by $\langle u, \omega \rangle_\star = \langle \bar{f}^\alpha(u), \bar{f}_\alpha(\omega) \rangle$, (see Section 5 for a full discussion of the pairing and of equations (4.34)-(4.36)). The connection ∇^* satisfies the Leibniz rule, for all functions h and vector fields v ,

$$\nabla^*(h \star v) = dh \otimes_\star v + h \star \nabla^*(v) .\tag{4.36}$$

Theorem 4. Consider a NC (pseudo-)Riemannian manifold M with metric g and abelian twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}$. Consider also the associated NC (pseudo-)Riemannian manifold M_{reg} and the NC Levi-Civita connection ∇^* on M_{reg} uniquely defined by the Christoffel symbols (2.13). This connection has a unique smooth extension $\hat{\nabla}^*$ to M . This extension is the NC Levi-Civita connection on M .

Proof. In Section 5.5, Theorem 5, we show that the torsion free and metric compatibility conditions uniquely determine the expression of $\hat{\nabla}^*$ on two arbitrary vector fields u and v . This is the case at zeroth order in the noncommutativity parameter θ , and the higher order θ -terms in $\hat{\nabla}^*$ can be perturbatively calculated. This shows the unicity of the connection $\hat{\nabla}^*$. It also shows that $\hat{\nabla}_u^*(v)$ is a smooth vector field if u and v are smooth, and that $\hat{\nabla}^*$ has local properties, i.e. if $u = u'$ and $v = v'$ in an open $U \in M$ then

$$\hat{\nabla}_u^*(v) = \hat{\nabla}_{u'}^*(v')\tag{4.37}$$

in $U \in M$. In order to prove that this map $\hat{\nabla}^*$ is a connection we have to show that it satisfies the Leibniz rule. We proceed in three steps.

- 1) It is easy to see that the covariant derivative ∇_μ^* on M_{reg} , uniquely defined by the Christoffel symbols (2.13), defines a connection ∇^* on M_{reg} ; indeed the Leibnitz rule for ∇^* immediately follows from the ∇_μ^* Leibniz rule (3.20).
- 2) In M_{reg} both $\hat{\nabla}^*$ and ∇^* are metric compatible and torsion free and therefore they coincide.
- 3) The unique smooth extension $\hat{\nabla}^*$ of ∇^* is a connection because the Leibnitz rule

$$\hat{\nabla}^*(h \star v) = dh \otimes_\star v + h \star \hat{\nabla}^*(v)\tag{4.38}$$

holds for any point $P \in M_{reg}$ and hence by continuity for any point $P \in M$. \square

Remark. Theorem 4 is the fundamental theorem of NC Riemannian Geometry with noncommutativity given by an arbitrary abelian twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}$.

Continuity of the NC Riemann, Ricci and metric tensors leads immediately to generalize to M the results found for M_{reg} in Theorem 2;

Corollary 1. Given a NC manifold M with abelian twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}$, consider all metrics \mathbf{g} such that the associated Killing Lie algebra g_K has the twist compatibility property,

$$\theta^{ab}X_a \otimes X_b \in \Xi \otimes g_K + g_K \otimes \Xi.$$

i) The NC Riemann tensors and the NC Ricci tensors of the NC Levi-Civita connections of these metrics \mathbf{g} are the undeformed ones.

ii) If these metrics \mathbf{g} are Einstein metrics then they are also NC Einstein metrics. \square

5 Drinfeld Twist Differential Geometry

In this section we consider a general (triangular) Drinfeld twist on a manifold M and construct the corresponding noncommutative differential and Riemannian geometry. The results obtained, in particular those regarding the properties of the contraction operator and pairing between covariant and contravariant tensors, and the uniqueness and smoothness of the NC Levi Civita connection, are needed in order to construct explicit NC differential geometry solutions.

5.1 Deformation by twists

We begin by recalling the general setting used to introduce a star product via a twist.

Consider a Lie algebra g over \mathbb{C} , and its associated universal enveloping algebra Ug . We recall that the elements of Ug are the complex numbers \mathbb{C} and sums of products of elements $t \in g$, where we identify $tt' - t't$ with the Lie algebra element $[t, t']$. Ug is an associative algebra with unit. It is a Hopf algebra with coproduct $\Delta : Ug \rightarrow Ug \otimes Ug$, counit $\varepsilon : Ug \rightarrow \mathbb{C}$ and antipode S given on the generators as:

$$\Delta(t) = t \otimes 1 + 1 \otimes t \quad \Delta(1) = 1 \otimes 1 \quad (5.39)$$

$$\varepsilon(t) = 0 \quad \varepsilon(1) = 1 \quad (5.40)$$

$$S(t) = -t \quad S(1) = 1 \quad (5.41)$$

and extended to all $U(g)$ by requiring Δ and ε to be linear and multiplicative (e.g. $\Delta(tt') := \Delta(t)\Delta(t') = tt' \otimes 1 + t \otimes t' + t' \otimes t + 1 \otimes tt'$), while S is linear and antimultiplicative.

In the sequel we use Sweedler coproduct notation

$$\Delta(\xi) = \xi_1 \otimes \xi_2 \quad (5.42)$$

where $\xi \in Ug$, $\xi_1 \otimes \xi_2 \in Ug \otimes Ug$ and a sum over ξ_1 and ξ_2 is understood.

We extend the notion of enveloping algebra to formal power series in λ (we replace the field \mathbb{C} with the ring $\mathbb{C}[[\lambda]]$) and we correspondingly consider the Hopf algebra $(Ug[[\lambda]], \cdot, \Delta, S, \varepsilon)$. In the sequel for sake of brevity we denote $Ug[[\lambda]]$ by Ug .

A twist \mathcal{F} is an element $\mathcal{F} \in Ug \otimes Ug$ that is invertible and that satisfies

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}, \quad (5.43)$$

$$(\varepsilon \otimes id)\mathcal{F} = 1 = (id \otimes \varepsilon)\mathcal{F}, \quad (5.44)$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$.

We in addition require⁴

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\lambda). \quad (5.45)$$

Property (5.43) states that \mathcal{F} is a two cocycle, this property is responsible for the associativity of the \star -products to be defined. Property (5.44) is just a normalization condition. From (5.45) it follows that \mathcal{F} can be formally inverted as a power series in λ . It also shows that the geometry we are going to construct has the nature of a deformation, i.e. in the 0-th order in λ we recover the usual undeformed geometry.

In the notation (sum over α understood)

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha, \quad (5.46)$$

the elements $f^\alpha, f_\alpha, \bar{f}^\alpha, \bar{f}_\alpha$ belong to Ug .

In order to become more familiar with this notation we rewrite equation (5.43) and its inverse,

$$((\Delta \otimes id)\mathcal{F}^{-1})\mathcal{F}_{12}^{-1} = ((id \otimes \Delta)\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}, \quad (5.47)$$

as well as (5.44) and (5.45) using the notation (5.46):

$$f^\beta f_1^\alpha \otimes f_\beta f_2^\alpha \otimes f_\alpha = f^\alpha \otimes f^\beta f_{\alpha_1} \otimes f_\beta f_{\alpha_2}, \quad (5.48)$$

$$\bar{f}_1^\alpha \bar{f}^\beta \otimes \bar{f}_2^\alpha \bar{f}_\beta \otimes \bar{f}_\alpha = \bar{f}^\alpha \otimes \bar{f}_{\alpha_1} \bar{f}^\beta \otimes \bar{f}_{\alpha_2} \bar{f}_\beta, \quad (5.49)$$

$$\varepsilon(f^\alpha) f_\alpha = 1 = f^\alpha \varepsilon(f_\alpha), \quad (5.50)$$

$$\mathcal{F} = f^\alpha \otimes f_\alpha = 1 \otimes 1 + \mathcal{O}(\lambda). \quad (5.51)$$

⁴Actually it is possible to show that (5.45) is a consequence of (5.43), (5.44) and of \mathcal{F} being at each order in λ a finite sum of finite products of Lie algebra elements

Consider now an algebra A (over $\mathbb{C}[[\lambda]]$), and an action of the Lie algebra g on A , $a \mapsto t(a)$ where $t \in g$ and $a \in A$. We require compatibility of this action with the product in A i.e. for any $t \in g$ we have a derivation of A ,

$$t(ab) = t(a)b + at(b) . \quad (5.52)$$

The action of g on A induces an action of the universal enveloping algebra Ug on A (for example the element $tt' \in Ug$ has action $t(t'(a))$). We say that A is a Ug -module algebra, i.e., the algebra structure of the Ug -module A is compatible with the Ug action, for all $\xi \in U\Xi$ and $a, b \in A$,

$$\xi(ab) = \mu \circ \Delta(\xi)(a \otimes b) = \xi_1(a)\xi_2(b) \quad , \quad \xi(1) = \varepsilon(\xi)1 . \quad (5.53)$$

(where 1 is the unit in A). This property is equivalent to (5.52).

Given a twist $\mathcal{F} \in Ug \otimes Ug$, we can construct a deformed algebra A_\star . The algebra A_\star has the same vector space structure as A . The product in A_\star is defined by

$$a \star b = \mu \circ \mathcal{F}^{-1}(a \otimes b) = \bar{f}^\alpha(a)\bar{f}_\alpha(b) . \quad (5.54)$$

In order to prove associativity of the new product we use (5.49) and compute:

$$\begin{aligned} (a \star b) \star c &= \bar{f}^\alpha(\bar{f}^\beta(a)\bar{f}_\beta(b))\bar{f}_\alpha(c) = (\bar{f}_1^\alpha\bar{f}^\beta)(a)(\bar{f}_2^\alpha\bar{f}_\beta)(b)\bar{f}_\alpha(c) = \bar{f}^\alpha(a)(\bar{f}_{\alpha_1}\bar{f}^\beta)(b)(\bar{f}_{\alpha_2}\bar{f}_\beta)(c) \\ &= \bar{f}^\alpha(a)\bar{f}_\alpha(\bar{f}^\beta(b)\bar{f}_\beta(c)) = a \star (b \star c) . \end{aligned}$$

Examples of twists include the Moyal-Weyl and the abelian twists of the previous sections. Twists however are not necessarily related to abelian Lie algebras. For example consider the elements H, E, A, B , that satisfy the Lie algebra relations

$$\begin{aligned} [H, E] &= 2E , \quad [H, A] = \alpha A , \quad [H, B] = \beta B , \quad \alpha + \beta = 2 , \\ [A, B] &= E , \quad [E, A] = 0 , \quad [E, B] = 0 . \end{aligned} \quad (5.55)$$

Then the element

$$\mathcal{F} = e^{\frac{1}{2}H \otimes \ln(1+\lambda E)} e^{\lambda A \otimes B \frac{1}{1+\lambda E}} \quad (5.56)$$

is a twist and gives a well defined \star -product on the algebra of functions on M . When $\lambda \rightarrow 0$ we recover the undeformed product. These twists are known as extended Jordanian deformations [25]. Jordanian deformations [23, 24] are obtained setting $A = B = 0$ (and keeping the relation $[H, E] = 2E$).

5.2 \star -Noncommutative Manifolds

A \star -noncommutative manifold is a smooth manifold M with a twist $\mathcal{F} \in U\Xi \otimes U\Xi$, where Ξ is the Lie algebra of vector fields on M .

We now use the twist to deform the commutative geometry on a manifold M (vector fields, 1-forms, exterior algebra, tensor algebra, symmetry algebras, covariant derivatives etc.) into the twisted noncommutative one. The guiding principle is the observation that every time we have a bilinear map

$$\mu : X \times Y \rightarrow Z$$

where X, Y, Z are vector spaces, and where there is an action of the Lie algebra g (and therefore of \mathcal{F}^{-1}) on X and Y we can compose this map with the action of the twist. In this way we obtain a deformed version μ_\star of the initial bilinear map μ :

$$\mu_\star := \mu \circ \mathcal{F}^{-1} , \quad (5.57)$$

$$\begin{aligned} \mu_\star : X \times Y &\rightarrow Z \\ (x, y) &\mapsto \mu_\star(x, y) = \mu(\bar{f}^\alpha(x), \bar{f}_\alpha(y)) . \end{aligned}$$

The action of the Lie algebra Ξ of vector fields on the vector spaces X, Y, Z we consider will always be via the Lie derivative.

Using (5.57), noncommutative functions A_\star , vector fields Ξ_\star and tensor fields \mathcal{T}_\star have been defined in Section 2.

We next introduce the universal \mathcal{R} -matrix

$$\mathcal{R} := \mathcal{F}_{21} \mathcal{F}^{-1} \quad (5.58)$$

where by definition $\mathcal{F}_{21} = f_\alpha \otimes f^\alpha$. In the sequel we use the notation

$$\mathcal{R} = R^\alpha \otimes R_\alpha , \quad \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha . \quad (5.59)$$

The \mathcal{R} -matrix measures the noncommutativity of the \star -product. Indeed it is easy to see that

$$h \star g = \bar{R}^\alpha(g) \star \bar{R}_\alpha(h) . \quad (5.60)$$

The permutation group in noncommutative space is naturally represented by \mathcal{R} . Formula (5.60) says that the \star -product is \mathcal{R} -commutative .

Exterior forms $\Omega_\star^\bullet = \oplus_p \Omega_\star^p$. Exterior forms form an algebra with product $\wedge : \Omega^\bullet \times \Omega^\bullet \rightarrow \Omega^\bullet$. We \star -deform the wedge product into the \star -wedge product,

$$\vartheta \wedge_\star \vartheta' := \bar{f}^\alpha(\vartheta) \wedge \bar{f}_\alpha(\vartheta') . \quad (5.61)$$

We denote by Ω_\star the linear space of forms equipped with the wedge product \wedge_\star .

As in the commutative case exterior forms are totally \star -antisymmetric contravariant tensor fields. For example the 2-form $\omega \wedge_\star \omega'$ is the \star -antisymmetric combination

$$\omega \wedge_\star \omega' = \omega \otimes_\star \omega' - \bar{R}^\alpha(\omega') \otimes_\star \bar{R}_\alpha(\omega) . \quad (5.62)$$

The exterior derivative $d : A \rightarrow \Omega$ satisfies the Leibniz rule $d(h \star g) = dh \star g + h \star dg$, indeed

$$\begin{aligned} d(h \star g) &= d(\bar{f}^\alpha(h) \bar{f}_\alpha(g)) = d(\bar{f}^\alpha(h)) \bar{f}_\alpha(g) + \bar{f}^\alpha(h) d\bar{f}_\alpha(g) \\ &= \bar{f}^\alpha(dh) \bar{f}_\alpha(g) + \bar{f}^\alpha(h) \bar{f}_\alpha(dg) \\ &= dh \star g + h \star dg \end{aligned} \quad (5.63)$$

where we observed that for each index α , \bar{f}^α and \bar{f}_α are products of vector fields acting via the Lie derivative on functions or 1-forms, and that therefore commute with the exterior differential because $\mathcal{L}_{uv\dots z} \equiv \mathcal{L}_u \circ \mathcal{L}_v \circ \dots \mathcal{L}_z$ and the Lie derivative along a vector field commutes with the differential.

The usual exterior derivative is therefore also the \star -exterior derivative.

\star -Pairing. We now consider the bilinear map $\langle \cdot, \cdot \rangle : \Xi \times \Omega \rightarrow A$, $(v, \omega) \mapsto \langle v, \omega \rangle$, where, using local coordinates, $\langle v^\mu \partial_\mu, \omega_\nu dx^\nu \rangle = v^\mu \omega_\mu$. Always according to the general prescription (5.57) we deform this pairing into

$$\langle \cdot, \cdot \rangle_\star : \Xi_\star \times \Omega_\star \rightarrow A_\star , \quad (5.64)$$

$$(\xi, \omega) \mapsto \langle \xi, \omega \rangle_\star := \langle \bar{f}^\alpha(\xi), \bar{f}_\alpha(\omega) \rangle . \quad (5.65)$$

It is easy to see that due to the cocycle condition for \mathcal{F} the \star -pairing satisfies the A_\star -linearity properties

$$\langle h \star u, \omega \star k \rangle_\star = h \star \langle u, \omega \rangle_\star \star k , \quad (5.66)$$

$$\langle u, h \star \omega \rangle_\star = \bar{R}^\alpha(h) \star \langle \bar{R}_\alpha(u), \omega \rangle_\star . \quad (5.67)$$

Using the pairing $\langle \cdot, \cdot \rangle_\star$ we associate with any 1-form ω the left A_\star -linear map $\langle \cdot, \omega \rangle_\star$. Also the converse holds: any left A_\star -linear map $\Phi : \Xi_\star \rightarrow A_\star$ is of the form $\langle \cdot, \omega \rangle_\star$ for some ω .

The pairing can be extended to covariant tensors and contravariant ones. We first define in the undeformed case the pairing (u^i denote vector fields, θ^j denote 1-forms),

$$\langle u^p \dots \otimes u^2 \otimes u^1, \theta^1 \otimes \theta^2 \dots \otimes \theta^p \rangle = \langle u^1, \theta^1 \rangle \langle u^2, \theta^2 \rangle \dots \langle u^p, \theta^p \rangle \quad (5.68)$$

and more in general the pairing

$$\langle u^p \dots \otimes u^1, \theta^1 \otimes \dots \otimes \theta^p \otimes \tau \rangle = \langle u^1, \theta^1 \rangle \dots \langle u^p, \theta^p \rangle \tau \quad (5.69)$$

that is obtained by first contracting the innermost elements; here τ is an arbitrary tensor field. Using locality and linearity this pairing is extended to any p -covariant tensor $\nu \in \mathcal{T}^{0,p}$ and any tensor $\rho \in \mathcal{T}^{q,s}$ at least p -times contravariant ($q \geq p$). It is this onion-like structure pairing that naturally generalizes to the noncommutative case.

The \star -pairing is defined by

$$\langle \nu, \rho \rangle_\star := \langle \bar{f}^\alpha(\nu), \bar{f}_\alpha(\rho) \rangle . \quad (5.70)$$

Using the cocycle condition for the twist \mathcal{F} and the onion like structure of the undeformed pairing we have the property

$$\langle \nu \otimes_\star u, \rho \rangle_\star = \langle \nu, \langle u, \rho \rangle_\star \rangle_\star . \quad (5.71)$$

\star -Lie algebra of vector fields Ξ_\star . The Lie algebra of vector fields is the Lie algebra of infinitesimal transformations (infinitesimal local diffeomorphisms). Vector fields act on tensor fields via the Lie derivative. The relativity principle of general covariance is implemented as general covariance under infinitesimal diffeomorphisms. These are given by Lie derivatives. In the noncommutative case the Lie algebra of vector fields is deformed, and applying the recipe (5.57) we obtain the \star -Lie bracket

$$\begin{aligned} [\]_\star : \quad \Xi \times \Xi &\rightarrow \Xi \\ (u, v) &\mapsto [u, v]_\star := [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] . \end{aligned} \quad (5.72)$$

This can be realized as a deformed commutator

$$\begin{aligned} [u, v]_\star &= [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] = \bar{f}^\alpha(u) \bar{f}_\alpha(v) - \bar{f}_\alpha(v) \bar{f}^\alpha(u) \\ &= u \star v - \bar{R}^\alpha(v) \star \bar{R}_\alpha(u) , \end{aligned} \quad (5.73)$$

where the \star -product between vector fields is given by $u \star v = \bar{f}^\alpha(u) \bar{f}_\alpha(v)$.

It is easy to see that the bracket $[\]_\star$ has the \star -antisymmetry property

$$[u, v]_\star = -[\bar{R}^\alpha(v), \bar{R}_\alpha(u)]_\star . \quad (5.74)$$

This can be shown as follows: $[u, v]_\star = [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] = -[\bar{f}_\alpha(v), \bar{f}^\alpha(u)] = -[\bar{R}^\alpha(v), \bar{R}_\alpha(u)]_\star$. A \star -Jacoby identity can be proven as well

$$[u, [v, z]_\star]_\star = [[u, v]_\star, z]_\star + [\bar{R}^\alpha(v), [\bar{R}_\alpha(u), z]_\star]_\star . \quad (5.75)$$

In the commutative case the commutator $[u, v]$ equals the Lie derivative $\mathcal{L}_u(v)$. It is then natural to define the \star -Lie derivative as

$$\mathcal{L}_u^\star := \mathcal{L}_{\bar{f}^\alpha(u)} \circ \bar{f}^\alpha , \quad (5.76)$$

so that $\mathcal{L}_u^\star(v) = [u, v]_\star$. Notice that the \star -Lie derivative is given by combining the usual Lie derivative with the twist \mathcal{F} as in (5.57).

Definition (5.76) holds more in general when the \star -Lie derivative acts on tensor fields \mathcal{T}_\star . The \star -Lie derivative satisfies the deformed Leibniz rule, for all $\tau, \tau' \in \mathcal{T}_\star$,

$$\mathcal{L}_u^\star(\tau \otimes_\star \tau') = \mathcal{L}_u^\star(\tau) \otimes_\star \tau' + \bar{R}^\alpha(\tau) \otimes_\star \mathcal{L}_{\bar{R}_\alpha(u)}^\star(\tau') ; \quad (5.77)$$

in particular on functions h, g we have $\mathcal{L}_u^\star(h \star g) = \mathcal{L}_u^\star(h) \star g + \bar{R}^\alpha(h) \star \mathcal{L}_{\bar{R}_\alpha(u)}^\star(g)$.

5.3 Covariant Derivative

A connection is a linear mapping

$$\nabla^\star : \Xi_\star \rightarrow \Omega_\star \otimes_\star \Xi_\star \quad (5.78)$$

which satisfies the (undeformed) Leibniz rule, for all $v \in \Xi_\star$,

$$\nabla^\star(h \star v) = dh \otimes_\star v + h \star \nabla^\star(v) . \quad (5.79)$$

Associated with a connection ∇^\star we have the covariant derivative ∇_u^\star along the vector field $u \in \Xi_\star$. It is defined by, for all $v \in \Xi_\star$

$$\nabla_u^\star(v) := \langle u, \nabla^\star v \rangle_\star . \quad (5.80)$$

From (5.79) and (5.80) we immediately have, for all $u, v, z \in \Xi_\star$, $h \in A_\star$,

$$\nabla_{u+v}^\star z = \nabla_u^\star z + \nabla_v^\star z , \quad (5.81)$$

$$\nabla_{h \star u}^\star v = h \star \nabla_u^\star v , \quad (5.82)$$

$$\nabla_u^\star(v + z) = \nabla_u^\star(v) + \nabla_u^\star(z) , \quad (5.83)$$

$$\nabla_u^\star(h \star v) = \mathcal{L}_u^\star(h) \star v + \bar{R}^\alpha(h) \star \nabla_{\bar{R}_\alpha(u)}^\star v . \quad (5.84)$$

We notice that the covariant derivative ∇_u^\star satisfies the same deformed Leibniz rule as the Lie derivative \mathcal{L}_u^\star (cf. (5.77)). As in the undeformed case we define the covariant derivative on functions to be equal to the Lie derivative, for all $h \in A_\star$,

$$\nabla_u^\star(h) = \mathcal{L}_u^\star(h) . \quad (5.85)$$

We also notice that the covariant derivative is defined only along vector fields and not along products of vector fields. The right hand side of expression (5.84) is well defined because of the peculiar property of the Leibniz rule (5.77): $\bar{R}_\alpha(u)$ is again a vector field.

With respect to a local frame of vector fields $\{e_i\}$ we have the connection coefficients

$$\nabla_{e_i}^* e_j = \Gamma_{ij}^k \star e_k . \quad (5.86)$$

Covariant derivative on tensor fields. We define the covariant derivative on covariant tensors by iterated use of the following deformed Leibniz rule [26], for all $u, v, z \in \Xi_\star$,

$$\nabla_u^*(v \otimes_\star z) := \bar{R}^\alpha(\nabla_{\bar{R}_\beta(u)}^* \bar{R}_\gamma(v)) \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z) + \bar{R}^\alpha(v) \otimes_\star \nabla_{\bar{R}_\alpha(u)}^* z . \quad (5.87)$$

As in the commutative case we define the covariant derivative on 1-forms Ω_\star by requiring compatibility with the contraction operator, for all $u, v \in \Xi_\star, \omega \in \Omega_\star$,

$$\nabla_u^* \langle v, \omega \rangle_\star = \langle \bar{R}^\alpha(\nabla_{\bar{R}_\beta(u)}^* \bar{R}_\gamma(v)), \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(\omega) \rangle_\star + \langle \bar{R}^\alpha(v), \nabla_{\bar{R}_\alpha(u)}^* \omega \rangle_\star \quad (5.88)$$

so that $\langle v, \nabla_u^* \omega \rangle_\star = \mathcal{L}_{\bar{R}^\alpha(u)}^* \langle \bar{R}_\alpha(v), \omega \rangle_\star - \langle \bar{R}^\alpha(\nabla_{\bar{R}_\beta \bar{R}^\delta(u)}^* \bar{R}_\gamma \bar{R}_\delta(v)), \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(\omega) \rangle_\star$. Finally we extend the covariant derivative to all tensor fields via the deformed Leibniz rule (5.87) where now $\tau, \tau' \in \mathcal{T}_\star$,

$$\nabla_u^*(\tau \otimes_\star \tau') := \bar{R}^\alpha(\nabla_{\bar{R}_\beta(u)}^* \bar{R}_\gamma(\tau)) \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(\tau') + \bar{R}^\alpha(\tau) \otimes_\star \nabla_{\bar{R}_\alpha(u)}^* \tau' . \quad (5.89)$$

5.4 Torsion, Curvature and Ricci tensor

The torsion T^* and the curvature R^* associated with a connection ∇^* are defined by

$$\mathsf{T}^*(u, v) := \nabla_u^* v - \nabla_{\bar{R}^\alpha(u)}^* \bar{R}_\alpha(v) - [u, v]_\star , \quad (5.90)$$

$$\mathsf{R}^*(u, v, z) := \nabla_u^* \nabla_v^* z - \nabla_{\bar{R}^\alpha(u)}^* \nabla_{\bar{R}_\alpha(v)}^* z - \nabla_{[u, v]_\star}^* z , \quad (5.91)$$

for all $u, v, z \in \Xi_\star$.

The presence of the R -matrix in the definition of torsion and curvature ensures that T^* and R^* are left A_\star -linear maps, i.e.

$$\mathsf{T}^*(f \star u, v) = f \star \mathsf{T}^*(u, v) \quad , \quad \mathsf{T}^*(u, f \star v) = \bar{R}^\alpha(f) \star \mathsf{T}^*(\bar{R}_\alpha(u), v)$$

and similarly for the curvature. The A_\star -linearity of T^* and R^* ensures that we have a well defined torsion tensor and curvature tensor. We denote them by the same letters T^* and R^* ; they are given by, for all $u, v \in \Xi_\star, \omega \in \Omega_\star$,

$$\langle u \otimes_\star v, \mathsf{T}^* \rangle_\star = \mathsf{T}^*(u, v) , \quad (5.92)$$

$$\langle u \otimes_\star v \otimes_\star z, \mathsf{R}^* \rangle_\star = \mathsf{R}^*(u, v, z) . \quad (5.93)$$

Local coordinates description. We denote by $\{e_i\}$ a local frame of vector fields (subordinate to an open $U \subset M$) and by $\{\theta^j\}$ the dual frame of 1-forms:

$$\langle e_i, \theta^j \rangle_\star = \delta_i^j . \quad (5.94)$$

The coefficients T_{ij}^{*l} and R_{ijk}^{*l} of the torsion and curvature tensors with respect to this local frame are uniquely defined by the following expressions

$$T^* = \theta^j \otimes_\star \theta^i \star T_{ij}^{*l} \otimes_\star e_l , \quad (5.95)$$

$$R^* = \theta^k \otimes_\star \theta^j \otimes_\star \theta^i \star R_{ijk}^{*l} \otimes_\star e_l , \quad (5.96)$$

so that $T_{ij}^{*l} = \langle T^*(e_i, e_j), \theta^l \rangle_\star$, $R_{ijk}^{*l} = \langle R^*(e_i, e_j, e_k), \theta^l \rangle_\star$. We also have [20]

$$T^* = \frac{1}{2} \theta^j \wedge_\star \theta^i \star T_{ij}^{*l} \otimes_\star e_l , \quad (5.97)$$

$$R^* = \frac{1}{2} \theta^k \otimes_\star \theta^j \wedge_\star \theta^i \star R_{ijk}^{*l} \otimes_\star e_l . \quad (5.98)$$

We recall that the Ricci tensor is given by the following contraction of the curvature:

$$\text{Ric}^*(u, v) := \langle \theta^i, R^*(e_i, u, v) \rangle_\star , \quad (5.99)$$

where sum over i is understood. The contraction \langle , \rangle_\star in (5.99) is a contraction between forms on the *left* and vector fields on the *right*. It is defined through the deformation of the commutative pairing, $\langle \omega , u \rangle_\star = \langle \bar{f}^\alpha(\omega) , \bar{f}_\alpha(u) \rangle$.

Definition (5.99) is well given because it is independent from the choice of the frame $\{e_i\}$ (and the dual frame $\{\theta^i\}$), and because the Ricci map so defined is an A_\star -linear map.

The coefficients of the Ricci tensor are

$$\text{Ric}_{jk}^* = \text{Ric}^*(e_j, e_k) . \quad (5.100)$$

In the commutative limit these tensors become the usual torsion, curvature and Ricci tensors,

$$T^* \rightarrow T , \quad R^* \rightarrow R , \quad \text{Ric}^* \rightarrow \text{Ric} . \quad (5.101)$$

and in particular we recover the usual components relation $\text{Ric}_{jk} = R_{ijk}^k$.

5.5 \star -Riemannian geometry

Along these lines one can also consider \star -Riemannian geometry. In order to define a \star -metric we need to define \star -symmetric elements in $\Omega_\star \otimes_\star \Omega_\star$ where Ω_\star is the space of 1-forms. Recalling that permutations are implemented with the R -matrix we see that \star -symmetric elements are of the form

$$\omega \otimes_\star \omega' + \bar{R}^\alpha(\omega') \otimes_\star \bar{R}_\alpha(\omega) . \quad (5.102)$$

In particular any symmetric tensor in $\Omega \otimes \Omega$ is also a \star -symmetric tensor in $\Omega_\star \otimes_\star \Omega_\star$, indeed expansion of (5.102) gives $\bar{f}^\alpha(\omega) \otimes \bar{f}_\alpha(\omega') + \bar{f}_\alpha(\omega') \otimes \bar{f}^\alpha(\omega)$ that is a sum (over α) of symmetric tensors. Similarly for antisymmetric tensors.

In particular, since a commutative metric is a nondegenerate symmetric tensor in $\Omega \otimes \Omega$ we conclude that any commutative metric is also a noncommutative metric, (\star -nondegeneracy of the metric is insured by the fact that at zeroth order in the deformation parameter the metric is nondegenerate). We denote by \mathbf{g} the metric tensor. If we write

$$\mathbf{g} = \mathbf{g}^a \otimes_\star \mathbf{g}_a \in \Omega_\star \otimes_\star \Omega_\star \quad (5.103)$$

(for example locally $\mathbf{g} = \theta^j \otimes_\star \theta^i \star \mathbf{g}_{ij}$), then for every $v \in \Xi_\star$ we can define the 1-form

$$\langle v, \mathbf{g} \rangle_\star := \langle v, \mathbf{g}^a \rangle_\star \star \mathbf{g}_a \quad (5.104)$$

and we can then construct the left A_\star -linear map \mathbf{g} , corresponding to the metric tensor $\mathbf{g} \in \Omega_\star \otimes_\star \Omega_\star$, as

$$\begin{aligned} \mathbf{g} : \Xi_\star \otimes_\star \Xi_\star &\rightarrow A_\star \\ (u, v) &\mapsto \mathbf{g}(u, v) = \langle u \otimes_\star v, \mathbf{g} \rangle_\star := \langle u, \langle v, \mathbf{g} \rangle_\star \rangle_\star . \end{aligned} \quad (5.105)$$

The \star -inverse metric $\mathbf{g}^{-1} \in \Xi_\star \otimes_\star \Xi_\star$ has been defined in [5] by first considering the metric as a map from vector fields to 1-forms and then by defining the inverse metric as the inverse of this map. If we write $\mathbf{g} = \mathbf{g}^a \otimes_\star \mathbf{g}_a$ and

$$\mathbf{g}^{-1} = \mathbf{g}^{-1b} \otimes_\star \mathbf{g}_b^{-1} \in \Xi_\star \otimes_\star \Xi_\star \quad (5.106)$$

then \mathbf{g}^{-1} is defined by the condition, for all $\omega \in \Omega_\star$,

$$\langle \langle \omega, \mathbf{g}^{-1} \rangle_\star, \mathbf{g} \rangle_\star = \omega . \quad (5.107)$$

We now consider the connection that has vanishing torsion and that is metric compatible, for all $u \in \Xi_\star$,

$$\nabla_u^\star \mathbf{g} = 0 ; \quad (5.108)$$

equivalently (recall (5.87)) for all $u, v, z \in \Xi_\star$,

$$\nabla_u^\star \mathbf{g}(v, z) = \mathbf{g}(\bar{R}^\alpha(\nabla_{\bar{R}_\beta(u)}^\star \bar{R}_\gamma(v)), \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z)) + \mathbf{g}(\bar{R}^\alpha(v), \nabla_{\bar{R}_\alpha(u)}^\star z) . \quad (5.109)$$

For the θ -constant case the explicit expression of the connection is via its Christoffel symbols (2.13), cf. [5]. For the case the twist is compatible with the metric as in (6.127), the existence of the Levi-Civita connection ∇^\star is proven in Theorem 8. For the abelian twist case existence of the Levi-Civita connection ∇^\star is proven in Theorem 4 (that uses Theorem 5 below).

We can then define the scalar curvature \mathfrak{R}^\star with respect to this connection. It is given by

$$\mathfrak{R}^\star := \langle \mathbf{g}^{-1}, \text{Ric}^\star \rangle_\star . \quad (5.110)$$

Locally we write $\mathbf{g}^{-1} = \mathbf{g}^{ij\star} \star e_j \otimes_\star e_i$, and

$$\mathfrak{R}^\star = \mathbf{g}^{ij\star} \star \text{Ric}^\star_{ji} . \quad (5.111)$$

The Einstein tensor is then defined by

$$\mathbf{G}^\star := \text{Ric}^\star - \frac{1}{2} \mathbf{g} \star \mathfrak{R}^\star , \quad (5.112)$$

and Einstein equations in vacuum are

$$\mathbf{G}^\star = 0 , \quad (5.113)$$

or equivalently,⁵ $\text{Ric}^\star = 0$.

We also define a \star -Einstein manifold to be a \star -Riemannian manifold that satisfies the condition

$$\text{Ric}^\star = c \mathbf{g} \quad (5.114)$$

for some real constant c (equivalently we can require the Einstein tensor \mathbf{G}^\star to be proportional to the metric tensor \mathbf{g}).

We conclude this section by showing that if a NC Levi-Civita connection exists, then it is unique and can be determined by a perturbative expansion order by order in the noncommutativity parameter. If we are able to prove that this unique expression thus obtained satisfies the Leibniz rule $\nabla^\star(h \star v) = dh \otimes_\star v + h \star \nabla^\star v$ then (as for example we do with the hypotheses of Theorem 4) we have existence and uniqueness of the NC Levi-Civita connection.

Theorem 5. Given a NC manifold M with metric \mathbf{g} , there exists a unique map $\nabla^\star : \Xi_\star \times \Xi_\star \rightarrow \Xi_\star$ that satisfies the torsion free condition $T^\star(u, v) = 0$ and the condition

$$\mathcal{L}_u^\star \langle v \otimes z, \mathbf{g} \rangle_\star = \langle \bar{R}^\alpha(\nabla_{\bar{R}_\beta(u)}^\star \bar{R}_\gamma(v)) \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z), \mathbf{g} \rangle_\star + \langle \bar{R}^\alpha(v) \otimes_\star \nabla_{\bar{R}_\alpha(u)}^\star(z), \mathbf{g} \rangle_\star \quad (5.115)$$

for all $u, v, z \in \Xi_\star$. This map is smooth, in the sense that $\nabla_u^\star v$ is a smooth vector field if u and v are smooth. It is local in the sense that if $u = u'$ and $v = v'$ in an open $U \subset M$ then $\hat{\nabla}_u^\star(v) = \hat{\nabla}_{u'}^\star(v')$ in $U \subset M$.

Proof. As in the undeformed case, use the \star -symmetry of the metric (cf. (5.102)) and the torsion free condition $T^\star(u, v) = 0$ in order to rewrite (5.115) as

$$\begin{aligned} \mathcal{L}_u^\star \langle v \otimes z, \mathbf{g} \rangle_\star &= \langle \bar{R}^\beta \bar{R}^\gamma(z) \otimes_\star \nabla_{\bar{R}^\delta \bar{R}_\gamma(v)}^\star \bar{R}_\delta \bar{R}_\beta(u), \mathbf{g} \rangle_\star + \\ &\quad + \langle \bar{R}^\beta \bar{R}^\gamma(z) \otimes_\star ([\bar{R}_\beta(u), \bar{R}_\gamma(v)]_\star), \mathbf{g} \rangle_\star + \langle \bar{R}^\gamma(v) \otimes_\star \nabla_{\bar{R}_\gamma(u)}^\star z, \mathbf{g} \rangle_\star \end{aligned}$$

⁵ $\mathbf{G}^\star = 0$ implies $\langle \mathbf{g}^{-1}, \mathbf{G}^\star \rangle_\star = 0$, and $1 - \frac{1}{2} \langle \mathbf{g}^{-1}, \mathbf{g} \rangle_\star \neq 0$ as is easily seen in the commutative limit.

Then sum and subtract the similar expression obtained by considering respectively the (undeformed) cyclic permutations $u, v, z \rightarrow v, z, u$ and $u, v, z \rightarrow z, u, v$. If we expand the R -matrix as $R = 1 \otimes 1 + \mathcal{O}(\lambda)$, and the \star -Lie derivative as $\mathcal{L}_u^\star = \mathcal{L}_u + \mathcal{O}(\lambda)$, where λ is the deformation quantization parameter such that for $\lambda \rightarrow 0$ we have the undeformed product (for example for abelian twist λ appears in $\mathcal{F} = e^{-\frac{i}{2}\lambda\theta^{ab}X_a \otimes X_b}$, and we have always included λ in θ^{ab}), the result is

$$2\mathbf{g}(\nabla_u^\star v, z) = \mathcal{L}_u \mathbf{g}(v, z) + \mathcal{L}_v \mathbf{g}(z, u) - \mathcal{L}_z \mathbf{g}(u, v) + \mathbf{g}(v, [z, u]) + \mathbf{g}(z, [u, v]) - \mathbf{g}(u, [v, z]) + \lambda Fun(\nabla^\star, u, v, z, \mathbf{g}, \mathcal{F}) \quad (5.116)$$

where $Fun(\nabla^\star, u, v, z, \mathbf{g}, \mathcal{F})$ is in particular a function of the connection ∇^\star . Now expand ∇^\star in power series of the noncommutativity parameter λ , $\nabla^\star = \nabla + \lambda \nabla_1 + \lambda^2 \nabla_2 \dots$. The n^{th} -order term ∇_n will depend, via (5.116), from the Lie derivative, u, v, z, \mathbf{g} and $\nabla_1, \dots, \nabla_{n-1}$. Thus (5.116) determines order by order in λ the NC map $\nabla_u^\star v$ for any u, v . Smoothness and locality properties are also immediate. \square

6 Geometry and Gravity Solutions III: Affine Killing Twists

In this section, as in the previous section, we consider a general twist \mathcal{F} on a manifold M , and we study connections that are compatible with \mathcal{F} . Their curvature and torsion tensors are undeformed. This way we arrive at gravity solutions associated to a general twist, not necessarily of the abelian kind.

6.1 Killing and affine Killing vector fields

Let us consider a (pseudo-)Riemannian manifold M with metric \mathbf{g} . A Killing vector field K is a vector field that leaves invariant the metric tensor,

$$\mathcal{L}_K \mathbf{g} = 0, \quad (6.117)$$

i.e., for any $u, v \in \Xi$

$$\mathcal{L}_K \mathbf{g}(u, v) = \mathbf{g}(\mathcal{L}_K(u), v) + \mathbf{g}(u, \mathcal{L}_K(v)). \quad (6.118)$$

Equivalently the (local) 1-parameter group of diffeomorphisms associated with the vector field K consists of (local) isometries.

The Lie bracket of two Killing vector fields is again a Killing vector field (indeed $\mathcal{L}_{[K, K']}\mathbf{g} = \mathcal{L}_K \mathcal{L}_{K'}\mathbf{g} - \mathcal{L}_{K'} \mathcal{L}_K \mathbf{g} = 0$). We thus have the Lie algebra of Killing vector fields.

Let us now consider the unique torsion free metric compatible connection ∇ associated with \mathbf{g} (the Levi-Civita or Riemannian connection). It is easy to prove that the Lie derivative along a Killing vector field of the covariant derivative ∇ vanishes:

$$\mathcal{L}_K \nabla = 0 , \quad (6.119)$$

i.e., for any $u, v \in \Xi^6$

$$\mathcal{L}_K(\nabla_u v) = \nabla_{[K, u]} v + \nabla_u \mathcal{L}_K v , \quad (6.121)$$

or equivalently, recalling that $\nabla_u v = \langle u, \nabla v \rangle$, for any $v \in \Xi$

$$\mathcal{L}_K(\nabla v) = \nabla \mathcal{L}_K v . \quad (6.122)$$

More generally we call a vector field K an affine Killing vector field of ∇ if it is compatible with the connection ∇ in the sense that (6.119) or (6.121) holds (here ∇ is an arbitrary connection not necessarily the Levi-Civita one). Geometrically the flux associated to K transforms parallel transported vector fields on a curve γ into parallel transported vector fields on the push forward of γ .

If we consider the Levi-Civita connection then a special class of infinitesimal Killing vector fields is provided by homothetic Killing vector fields, i.e. vector fields K that satisfy

$$\mathcal{L}_K \mathbf{g} = c \mathbf{g} \quad (6.123)$$

for some constant c (dependent on K). Homothetic Killing vector fields form a Lie algebra. It is easy to see that a homothetic Killing vector field is also an affine vector field (hint: apply the Lie derivative \mathcal{L}_K to equation (6.120)).

We denote by \mathbf{T} , \mathbf{R} and \mathbf{Ric} the commutative torsion, curvature and Ricci curvature associated with ∇ . If ∇ is the Levi-Civita connection of a (pseudo-)Riemannian manifold then we further denote by \mathbf{G} and \mathfrak{R} the Einstein tensor and the curvature scalar.

We recall that if K is an affine Killing vector then \mathbf{T} , \mathbf{R} and \mathbf{Ric} are invariant under K ,

$$\mathcal{L}_K \mathbf{T}(u, v) = \mathbf{T}(\mathcal{L}_K u, v) + \mathbf{T}(u, \mathcal{L}_K v) , \quad (6.124)$$

$$\mathcal{L}_K \mathbf{R}(u, v, z) = \mathbf{R}(\mathcal{L}_K u, v, z) + \mathbf{R}(u, \mathcal{L}_K v, z) + \mathbf{R}(u, v, \mathcal{L}_K z) , \quad (6.125)$$

$$\mathcal{L}_K \mathbf{Ric}(u, v) = \mathbf{Ric}(\mathcal{L}_K u, v) + \mathbf{Ric}(u, \mathcal{L}_K v) , \quad (6.126)$$

⁶*Proof.* Recall that the Levi-Civita connection is uniquely determined by the condition, for all $u, v, z \in \Xi$,

$$2\mathbf{g}(\nabla_u v, z) = \mathcal{L}_u \mathbf{g}(v, z) + \mathcal{L}_v \mathbf{g}(z, u) - \mathcal{L}_z \mathbf{g}(u, v) + \mathbf{g}(v, [z, u]) + \mathbf{g}(z, [u, v]) - \mathbf{g}(u, [v, z]) \quad (6.120)$$

(that using local coordinates x^i and the corresponding vector fields ∂_i reads $2\Gamma_{ij}^k \mathbf{g}_{kl} = \partial_i \mathbf{g}_{jk} + \partial_j \mathbf{g}_{ki} - \partial_k \mathbf{g}_{ij}$). Acting with the Killing vector field K on (6.120) and recalling (6.118) and writing $\mathcal{L}_K \mathcal{L}_u \mathbf{g}(v, z) = \mathcal{L}_{[K, u]} \mathbf{g}(v, z) + \mathcal{L}_u \mathbf{g}(\mathcal{L}_K v, z) + \mathcal{L}_u \mathbf{g}(v, \mathcal{L}_K z)$ we obtain (6.121).

(for a proof of this known result see e.g. [27], Chapter 6).

We have introduced the three Lie algebras of affine Killing vector fields, of homothetic Killing vector fields and of Killing vector fields. Depending on the Riemannian manifold we are considering these three notions can coincide. In particular we recall (see e.g. [27], Chapter 6) that in an irreducible Riemannian manifold (i.e., a manifold whose holonomy group acts irreducibly) every infinitesimal affine transformation is homothetic. Moreover on a compact Riemannian manifold every affine Killing vector field is a Killing vector field.

6.2 Affine Killing twists

We now consider a noncommutative deformation of a manifold M with connection ∇ . We study the case

$$\mathcal{F} \in U\hat{g}_K \otimes U\hat{g}_K , \quad (6.127)$$

where \hat{g}_K is the Lie algebra of affine Killing vectors of the connection ∇ . We thus relate the noncommutative structure of M to the symmetries of the linear connection ∇ of M .

Later on we consider ∇ to be the Levi-Civita connection of the (pseudo-)Riemannian manifold M with metric g .

Theorem 6. There is a canonical \star -connection ∇^\star associated to the \star -noncommutative manifold M with connection ∇ and compatible twist \mathcal{F} as in (6.127). The \star -connection ∇^\star is the undeformed connection ∇ itself,

$$\nabla^\star = \nabla , \quad (6.128)$$

where $\nabla^\star : \Xi_\star \rightarrow \Omega_\star \otimes_\star \Xi_\star$ while $\nabla : \Xi \rightarrow \Omega \otimes \Xi$ and we use that as vector spaces $\Xi_\star = \Xi$, $\Omega_\star = \Omega$ and $\Xi_\star \otimes_\star \Omega_\star = \Xi \otimes \Omega$.

The relation between the corresponding covariant derivatives is, for all $u \in \Xi_\star$,

$$\nabla_u^\star = \nabla_{\bar{f}^\alpha(u)} \circ \bar{f}_\alpha , \quad (6.129)$$

where $\nabla_u^\star : \mathcal{T}_\star \rightarrow \mathcal{T}_\star$, $\nabla_u : \mathcal{T} \rightarrow \mathcal{T}$ and we use that as vector spaces $\mathcal{T}_\star = \mathcal{T}$ and $\Xi_\star = \Xi$.

Proof. In order to show that ∇ is a noncommutative connection, i.e., in order to show that (5.79) holds, we use that the Lie derivative along an affine Killing vector commutes with the covariant derivative and the exterior derivative (cf. the derivation of (5.63)), and we recall that the action of the twist on tensors is via the Lie derivative. Therefore we have

$$\begin{aligned} \nabla(h \star v) &= \nabla(\bar{f}^\alpha(h) \bar{f}_\alpha(v)) = d\bar{f}^\alpha(h) \otimes \bar{f}_\alpha(v) + \bar{f}^\alpha(h) \nabla \bar{f}_\alpha(v) \\ &= \bar{f}^\alpha(dh) \otimes \bar{f}_\alpha(v) + \bar{f}^\alpha(h) \bar{f}_\alpha(\nabla v) \\ &= dh \otimes_\star v + h \star \nabla v . \end{aligned}$$

In order to prove relation (6.129) we first observe that this relation holds when ∇_u^* and $\nabla_{\bar{f}^\alpha(u)} \circ \bar{f}_\alpha$ act on functions, indeed on functions $\nabla_u^* = \mathcal{L}_u^* = \mathcal{L}_{\bar{f}^\alpha(u)}^* \circ \bar{f}_\alpha = \nabla_{\bar{f}^\alpha(u)} \circ \bar{f}_\alpha$. Similarly on vector fields,

$$\nabla_u^* v := \langle u, \nabla^* v \rangle_\star = \langle \bar{f}^\alpha(u), \bar{f}_\alpha(\nabla^* v) \rangle = \langle \bar{f}^\alpha(u), \nabla^* \bar{f}_\alpha(v) \rangle = \nabla_{\bar{f}^\alpha(u)} \bar{f}_\alpha(v) . \quad (6.130)$$

Next we show that $\nabla_{\bar{f}^\alpha(u)} \circ \bar{f}_\alpha$ satisfies the same deformed Leibniz rule as ∇_u^* . We notice that in $Ug \otimes Ug \otimes Ug$ we have

$$\begin{aligned} \bar{f}_2^\alpha \bar{f}_\beta \otimes \bar{f}_1^\alpha \bar{f}^\beta \otimes \bar{f}_\alpha &= \bar{f}_1^\alpha \bar{f}_\beta \otimes \bar{f}_2^\alpha \bar{f}^\beta \otimes \bar{f}_\alpha \\ &= \bar{f}_1^\alpha \bar{f}^\varepsilon f^\sigma \bar{f}_\beta \otimes \bar{f}_2^\alpha \bar{f}_\varepsilon f_\sigma \bar{f}^\beta \otimes \bar{f}_\alpha \\ &= \bar{f}^\alpha f^\sigma \bar{f}_\beta \otimes \bar{f}_{\alpha_1} \bar{f}^\varepsilon f_\sigma \bar{f}^\beta \otimes \bar{f}_{\alpha_2} \bar{f}_\varepsilon \\ &= \bar{f}^\alpha \bar{R}^\gamma \otimes \bar{f}_{\alpha_1} \bar{f}^\varepsilon \bar{R}_\gamma \otimes \bar{f}_{\alpha_2} \bar{f}_\varepsilon \end{aligned} \quad (6.131)$$

where in the first line we have used that the undeformed coproduct is cocommutative (and therefore $\bar{f}_2^\alpha \otimes \bar{f}_1^\alpha \otimes \bar{f}_\alpha = \bar{f}_1^\alpha \otimes \bar{f}_2^\alpha \otimes \bar{f}_\alpha$), in the second line we inserted the identity in the form $1 \otimes 1 = \mathcal{F}\mathcal{F}^{-1} = \bar{f}^\varepsilon f^\sigma \otimes \bar{f}_\varepsilon f_\sigma$, in the third line we used (5.49), and in the fourth line we recalled that $\mathcal{R}^{-1} = \mathcal{F}\mathcal{F}_{21}^{-1}$. We then compute

$$\begin{aligned} \nabla_{\bar{f}^\alpha(u)} \bar{f}_\alpha(v \otimes_\star z) &= \nabla_{\bar{f}^\alpha(u)} (\bar{f}_{\alpha_1} \bar{f}^\beta(v) \otimes \bar{f}_{\alpha_2} \bar{f}_\beta(z)) \\ &= \nabla_{\bar{f}_1^\alpha \bar{f}^\beta(u)} (\bar{f}_2^\alpha \bar{f}_\beta(v) \otimes \bar{f}_\alpha(z)) \\ &= \nabla_{\bar{f}_1^\alpha \bar{f}^\beta(u)} (\bar{f}_2^\alpha \bar{f}_\beta(v)) \otimes \bar{f}_\alpha(z) + \bar{f}_2^\alpha \bar{f}_\beta(v) \otimes \nabla_{\bar{f}_1^\alpha \bar{f}^\beta(u)} \bar{f}_\alpha(z) \\ &= \langle \bar{f}_1^\alpha \bar{f}^\beta(u), \bar{f}_2^\alpha \bar{f}_\beta(\nabla v) \rangle \otimes \bar{f}_\alpha(z) + \bar{f}^\alpha \bar{R}^\gamma(v) \otimes \langle \bar{f}_{\alpha_1} \bar{f}^\varepsilon \bar{R}_\gamma(u), \nabla \bar{f}_{\alpha_2} \bar{f}_\varepsilon(z) \rangle \\ &= \bar{f}^\alpha \langle u, \nabla v \rangle_\star \otimes \bar{f}_\alpha(z) + \bar{f}^\alpha \bar{R}^\gamma(v) \otimes \bar{f}_\alpha \langle \bar{R}_\gamma(u), \nabla z \rangle_\star \\ &= \langle u, \nabla v \rangle_\star \otimes_\star z + \bar{R}^\gamma(v) \otimes_\star \langle \bar{R}_\gamma(u), \nabla z \rangle_\star \\ &= \nabla_u^* v \otimes_\star z + \bar{R}^\gamma(v) \otimes_\star \nabla_{\bar{R}_\gamma(u)}^* z \end{aligned} \quad (6.132)$$

where in the second line we have used (5.49), and in the fourth line we recalled (6.131). This expression coincides with (5.87) for affine Killing twists. This can be seen by repeatedly applying (5.49) and cocommutativity of the undeformed coproduct, or also by considering the following equalities,

$$\begin{aligned} \bar{R}^\alpha(\nabla_{\bar{R}_\beta(u)}^* \bar{R}_\gamma(v)) \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z) &= \bar{R}^\alpha \langle \bar{R}_\beta(u), \nabla^* \bar{R}_\gamma(v) \rangle_\star \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z) \\ &= \langle \bar{R}_1^\alpha \bar{R}_\beta(u), \bar{R}_2^\alpha \nabla^* \bar{R}_\gamma(v) \rangle_\star \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z) \\ &= \langle \bar{R}_1^\alpha \bar{R}_\beta(u), \nabla^* \bar{R}_2^\alpha \bar{R}_\gamma(v) \rangle_\star \otimes_\star \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z) \\ &= \langle \bar{R}^\alpha \bar{R}_\beta(u), \nabla^* \bar{R}^\sigma \bar{R}_\gamma(v) \rangle_\star \otimes_\star \bar{R}_\sigma \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma(z) \\ &= \langle u, \nabla^* v \rangle_\star \otimes_\star z \\ &= \nabla_u^* v \otimes_\star z , \end{aligned} \quad (6.133)$$

where in the fourth line we used the R -matrix property $(\Delta \otimes_\star id)\mathcal{R}^{-1} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}$, i.e. $\bar{R}_1^\alpha \otimes_\star \bar{R}_2^\alpha \otimes_\star \bar{R}_\alpha = \bar{R}^\alpha \otimes_\star \bar{R}^\sigma \otimes_\star \bar{R}_\sigma \bar{R}_\alpha$, that follows from (5.49), and in the fifth line we used the triangularity property $\mathcal{R}_{12} = \mathcal{R}_{21}^{-1}$, that immediately follows from the definition of \mathcal{R} .

Since the Lie derivative and the covariant derivative commute with the contraction operator $\langle \cdot, \cdot \rangle$, we also have $\nabla_{\bar{f}^\alpha(u)} \bar{f}_\alpha \omega = \nabla_u^\star \omega$ for any 1-form (the proof is similar to (6.132) just consider $\langle v, \omega \rangle_\star$ rather than $v \otimes_\star z$; then recall (5.88)). This implies that the deformed Leibniz rule property (6.132) holds also if v and/or z are 1-forms. Iterated use of this property then shows that $\nabla_{\bar{f}^\alpha(u)} \circ \bar{f}_\alpha = \nabla_u^\star$ on any tensor field and for any vector field $u \in \Xi_\star$. \square

Theorem 7. The \star -torsion, \star -curvature and \star -Ricci tensors of the connection $\nabla^\star = \nabla$ are the undeformed ones,

$$\mathsf{T}^\star = \mathsf{T} \ , \quad \mathsf{R}^\star = \mathsf{R} \ , \quad \mathsf{Ric}^\star = \mathsf{Ric} \ . \quad (6.134)$$

These equalities holds when we consider $\mathsf{T}^\star, \mathsf{R}^\star, \mathsf{Ric}^\star$ as tensors, and we use that the noncommutative and commutative tensor spaces are equal as vector spaces, $\mathcal{T}_\star = \mathcal{T}$. When we consider $\mathsf{T}^\star, \mathsf{R}^\star, \mathsf{Ric}^\star$ as multilinear operators we have, due to invariance of $\mathsf{T}, \mathsf{R}, \mathsf{Ric}$ under affine Killing vector fields,

$$\langle u \otimes_\star v, \mathsf{T}^\star \rangle_\star = \langle u \otimes_\star v, \mathsf{T} \rangle \ , \quad \langle u \otimes_\star v \otimes_\star z, \mathsf{R}^\star \rangle_\star = \langle u \otimes_\star v \otimes_\star z, \mathsf{R} \rangle \ , \quad \langle u \otimes_\star v, \mathsf{Ric}^\star \rangle_\star = \langle u \otimes_\star v, \mathsf{Ric} \rangle$$

for all $u, v, z \in \Xi_\star$.

Proof. In order to prove that $\mathsf{T}^\star = \mathsf{T}$ we show that $\langle u \otimes_\star v, \mathsf{T}^\star \rangle_\star = \langle u \otimes_\star v, \mathsf{T} \rangle_\star$ for all $u, v \in \Xi_\star$,

$$\begin{aligned} \langle u \otimes_\star v, \mathsf{T}^\star \rangle_\star = \mathsf{T}^\star(u, v) &= \langle u, \nabla v \rangle_\star - \langle \bar{R}^\alpha(v), \nabla \bar{R}_\alpha(u) \rangle_\star - [u, v]_\star \\ &= \langle \bar{f}^\alpha(u), \bar{f}_\alpha(\nabla v) \rangle - \langle \bar{f}^\beta \bar{R}^\alpha(v), \bar{f}_\beta(\nabla \bar{R}_\alpha(u)) \rangle - [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] \\ &= \langle \bar{f}^\alpha(u), \nabla \bar{f}_\alpha(v) \rangle - \langle \bar{f}_\alpha(v), \nabla \bar{f}^\alpha(u) \rangle - [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] \\ &= \mathsf{T}(\bar{f}^\alpha(u), \bar{f}_\alpha(v)) \\ &= \langle u \otimes_\star v, \mathsf{T} \rangle \\ &= \langle u \otimes_\star v, \mathsf{T} \rangle_\star \end{aligned}$$

where we used the definition (5.58) of the R -matrix and, in the last equality, that the torsion tensor T is invariant under affine Killing vector fields.

We similarly prove $R^* = R$,

$$\begin{aligned}
\langle u \otimes_\star v \otimes_\star z, R^* \rangle_\star &= \nabla_u^* \nabla_v^* z - \nabla_{\bar{R}^\gamma(v)}^* \nabla_{\bar{R}_\gamma(u)}^* z - \nabla_{[u,v]_\star}^* z \\
&= \langle u, \nabla \langle v, \nabla z \rangle_\star \rangle_\star - \langle \bar{R}^\gamma(v), \nabla \langle \bar{R}_\gamma(u), \nabla z \rangle_\star \rangle_\star - \langle [u, v]_\star, \nabla z \rangle_\star \\
&= \langle \bar{f}_1^\alpha \bar{f}^\beta(u), \nabla \langle \bar{f}_2^\alpha \bar{f}_\beta(v), \nabla \bar{f}_\alpha(z) \rangle \rangle - \langle \bar{f}_2^\alpha \bar{f}_\beta(v), \nabla \langle \bar{f}_1^\alpha \bar{f}^\beta(u), \nabla \bar{f}_\alpha(z) \rangle \rangle \\
&\quad - \langle [\bar{f}_1^\alpha \bar{f}^\beta(u), \bar{f}_2^\alpha \bar{f}_\beta(v)], \nabla \bar{f}_\alpha(z) \rangle \\
&= \langle u \otimes_\star v \otimes_\star z, R \rangle \\
&= \langle u \otimes_\star v \otimes_\star z, R \rangle_\star
\end{aligned} \tag{6.135}$$

where in the third line we used (5.49) and (6.131).

In order to show that $\text{Ric}^* = \text{Ric}$ we write the identity operator on the space of 1-forms in two equivalent ways,

$$\text{id} = \check{\theta}^i \langle \check{e}_i, (\cdot) \rangle = \theta^i \star \langle e_i, (\cdot) \rangle_\star \tag{6.136}$$

where \check{e}_i and $\check{\theta}^i$ are dual bases while e_i and θ^i are \star -dual bases, $\langle \check{e}_i, \check{\theta}^j \rangle = \delta_i^j$, $\langle e_i, \theta^j \rangle_\star = \delta_i^j$. The equalities (6.136) follow immediately by decomposing a 1-form as $\omega = \check{\theta}^j \omega_j = \theta^j \star \omega_j^*$. These equalities imply that on any tensor $\tau \in \mathcal{T}^{1,1}$

$$\langle \theta^i, \langle e_i, \tau \rangle_\star \rangle_\star = \langle \check{\theta}^i, \langle \check{e}_i, \tau \rangle \rangle. \tag{6.137}$$

Indeed (locally) write $\tau = \tau^j \otimes_\star e_j$, where τ^j are 1-forms; it then follows

$$\begin{aligned}
\langle \theta^i, \langle e_i, \tau^j \otimes_\star e_j \rangle_\star \rangle_\star &= \langle \theta^i, \langle e_i, \tau^j \rangle_\star \star e_j \rangle_\star = \langle \theta^i \star \langle e_i, \tau^j \rangle_\star, e_j \rangle_\star = \langle \tau^j, e_j \rangle_\star = \langle \bar{f}^\alpha(\tau^j), \bar{f}_\alpha(e_j) \rangle \\
\langle \check{\theta}^i, \langle \check{e}_i, \tau^j \otimes_\star e_j \rangle \rangle &= \langle \check{\theta}^i, \langle \check{e}_i, \bar{f}^\alpha(\tau^j) \otimes \bar{f}_\alpha(e_j) \rangle \rangle = \langle \check{\theta}^i, \langle \check{e}_i, \bar{f}^\alpha(\tau^j) \rangle, \bar{f}_\alpha(e_j) \rangle = \langle \bar{f}^\alpha(\tau^j), \bar{f}_\alpha(e_j) \rangle
\end{aligned}$$

We then compute, for all $u, v \in \Xi_\star$,

$$\begin{aligned}
\langle u \otimes_\star v, \text{Ric}^* \rangle_\star &= \langle \theta^i, \langle e_i \otimes_\star u \otimes_\star v, R \rangle_\star \rangle_\star \\
&= \langle \theta^i, \langle e_i, \langle u \otimes_\star v, R \rangle_\star \rangle_\star \rangle_\star \\
&= \langle \check{\theta}^i, \langle \check{e}_i, \langle u \otimes_\star v, R \rangle_\star \rangle \rangle \\
&= \langle \check{\theta}^i, \langle \check{e}_i, \langle u \otimes_\star v, R \rangle \rangle \rangle \\
&= \langle \check{\theta}^i, \langle \check{e}_i \otimes \bar{f}^\alpha(u) \otimes \bar{f}_\alpha(v), R \rangle \rangle \\
&= \langle \bar{f}^\alpha(u) \otimes \bar{f}_\alpha(v), \text{Ric} \rangle \\
&= \langle u \otimes_\star v, \text{Ric} \rangle_\star
\end{aligned} \tag{6.138}$$

where in the second line we used the \star -pairing property (5.71), in the third line property (6.137) with $\tau = \langle u \otimes_\star v, R \rangle_\star$, and in the fourth and last lines the invariance of R and Ric under affine Killing vector fields. \square

6.3 Gravity solutions III

We now consider a (pseudo-)Riemannian manifold M with metric \mathbf{g} and associated Levi-Civita connection ∇ . If \mathbf{g} is positive definite (and if any two points of M can be connected by a geodesic) then we have the de Rham decomposition (see for ex. ref. [28]), $M = M_1 \times \dots \times M_p$, where M_1 is the Euclidean space \mathbb{R}^m , $m \geq 0$, and M_i , $i = 2, \dots, p$ are irreducible Riemannian manifolds not isometric to the real line. The metric \mathbf{g} is the direct sum of the standard Euclidean metric of \mathbb{R}^m and of the metrics \mathbf{g}_i of M_i . In this setting an affine Killing vector acts on each manifold M_i as an homothetic Killing vector ($\mathcal{L}_K \mathbf{g}_i = c_i \mathbf{g}_i$), see Theorem 3.6 in [27].

If M has Lorentzian (or more in general indefinite) signature, then, given a decomposition $M = M_1 \times \dots \times M_p$, we consider affine Killing vector fields K that when restricted to each M_i , $i = 1, 2, \dots, p$ act as homothetic Killing vector fields. These vector fields form a Lie subalgebra of the Lie algebra \hat{g}_K of affine Killing vector fields of M . We denote it by g_{hK} .

In this section we consider metrics \mathbf{g} that are compatible with the twist \mathcal{F} in the sense that

$$\mathcal{F} \in U g_{hK} \otimes U g_{hK} . \quad (6.139)$$

If this is the case we have

Theorem 8. The \star -Levi-Civita connection ∇^\star associated with the \star -noncommutative manifold M with twist \mathcal{F} and compatible metric \mathbf{g} as in (6.139) is the usual Levi-Civita connection of the commutative manifold M with metric \mathbf{g} ,

$$\nabla^\star = \nabla , \quad (6.140)$$

where $\nabla^\star : \Xi_\star \rightarrow \Omega_\star \otimes_\star \Xi_\star$ while $\nabla : \Xi \rightarrow \Omega \otimes \Xi$ and we use that as vector spaces $\Xi_\star = \Xi$, $\Omega_\star = \Omega$ and $\Xi_\star \otimes_\star \Omega_\star = \Xi \otimes \Omega$.

Proof. Because of Theorem 6 we just have to prove the compatibility of ∇^\star with the metric tensor. Let $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 + \dots + \mathbf{g}_p$ be the metric on $M = M_1 \times M_2 \times \dots \times M_p$, and $v = v^1 + v^2 + \dots + v^p$ a vector field. Then the Levi-Civita connection is the direct sum of the Levi-Civita connections on M_1, M_2, \dots, M_p , $\nabla_v = \nabla_{v^1}^1 + \nabla_{v^2}^2 + \dots + \nabla_{v^p}^p$. From the very definition of the Lie algebra g_{hK} and from (6.139) it follows that for each index α , $\bar{f}_\alpha \mathbf{g} = c_\alpha^1 \mathbf{g}_1 + c_\alpha^2 \mathbf{g}_2 + \dots + c_\alpha^p \mathbf{g}_p$ with $c_\alpha^1, c_\alpha^2, \dots, c_\alpha^p$ constant coefficients. Finally we have,

$$\nabla_u^\star \mathbf{g} = \nabla_{\bar{f}^\alpha(u)} \bar{f}_\alpha \mathbf{g} = \sum_{i=1}^p \nabla_{\bar{f}^\alpha(u)^i}^i c_\alpha^i \mathbf{g}_i = 0 . \quad (6.141)$$

□

We now apply Theorem 7 and conclude that the torsion, curvature and Ricci tensors of the connection $\nabla^* = \nabla$ are the undeformed ones. Therefore,

Corollary 2. If g is a commutative Einstein metric for the manifold M and (6.139) holds, then g is also a noncommutative Einstein metric. \square

For example let's consider the Connes-Landi 4-sphere [10]. It is obtained from an abelian Drinfeld twist constructed with Killing vector fields. It therefore satisfies condition (6.139). It is a noncommutative Einstein space with noncommutative connection, curvature and Ricci tensors equal to the undeformed ones.

Explicitly the usual 4-sphere is the subspace of \mathbb{R}^5 defined by $\sum_{i=1}^5 X^{i2} = 1$, or, using the complex coordinates $x^1 = (X^1 - iX^2)/\sqrt{2}$, $x^2 = (X^4 - iX^5)/\sqrt{2}$, $x^3 = X^3$, $x^4 = \overline{x^2}$, $x^5 = \overline{x^1}$, by $2x^1x^5 + 2x^2x^4 + x^3x^3 = 1$. The twist is

$$\mathcal{F} = e^{\frac{-i}{2}\lambda[(x^1\partial_1 - x^5\partial_5) \otimes (x^2\partial_2 - x^4\partial_4) - (x^2\partial_2 - x^4\partial_4) \otimes (x^1\partial_1 - x^5\partial_5)]} . \quad (6.142)$$

The x^i coordinates satisfy the noncommutative 4-sphere relations (cf. for example [15]):

$$\begin{aligned} 2x^1 \star x^5 + 2x^2 \star x^4 + x^3 \star x^3 &= 1 \quad , \quad x^3 \star x^i = x^i \star x^3 \quad (i = 1, 2, 4, 5) \\ x^1 \star x^2 &= qx^2 \star x^1 \quad , \quad x^1 \star x^4 = q^{-1}x^4 \star x^1 \quad , \quad x^1 \star x^5 = x^5 \star x^1 \quad , \\ x^2 \star x^5 &= qx^5 \star x^2 \quad , \quad x^4 \star x^5 = q^{-1}x^5 \star x^4 \quad , \quad x^2 \star x^4 = x^4 \star x^2 \quad , \end{aligned}$$

where $q = e^{i\lambda}$.

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